Nonlinear Resonant Vibrations of Infinitely Long Cylindrical Shells

JERRY H. GINSBERG*
Purdue University, Lafayette, Ind.

Theme

THIS paper examines the steady-state response of cylindrical shells to transverse excitations which would cause the shell to be close to resonance in a linearized analysis. The investigation is performed by a perturbation procedure which retains all generalized coordinates appearing in the first approximation of the effects of nonlinearity in the strain-displacement relations. Bending effects and tangential inertia effects are included, so that the results are applicable to a wide range of axial and circumferential wavelengths, and, for a given set of wavelengths, to excitation in the spectral neighborhood of any of the three lowest natural frequencies. Because the infinite shell lacks axial boundaries, the responses contain terms which are not allowable in the physically realistic problem of shells of finite length. However, the results for plane strain provide refinement of previous studies of rings.

Contents

The deformation of the shell is referred to the initial geometry. The displacements $\bar{u}$, $\bar{v}$, $\bar{w}$, in the axial, circumferential, and transverse directions, respectively, of a point at a distance z outward from the middle surface are related to the displacements, u, v, w of the middle surface by

$$\bar{u} = u - z\,\partial w/\partial x;\quad \bar{v} = v - (z/R)(\partial w/\partial\theta - v);\quad \bar{w} = w \tag{1}$$

The Lagrangian strains are

$$\begin{aligned}
\varepsilon_{xx} &= \partial\bar{u}/\partial x + \tfrac{1}{2}[(\partial\bar{u}/\partial x)^2 + (\partial\bar{v}/\partial x)^2 + (\partial\bar{w}/\partial x)^2] \\
\varepsilon_{\theta\theta} &= \frac{1}{(R+z)}\left(\frac{\partial\bar{v}}{\partial\theta} + \bar{w}\right) + \frac{1}{2(R+z)}\left[\left(\frac{\partial\bar{u}}{\partial\theta}\right)^2 + \left(\frac{\partial\bar{v}}{\partial\theta} + \bar{w}\right)^2 + \left(\frac{\partial\bar{w}}{\partial\theta} - \bar{v}\right)^2\right] \\
\varepsilon_{x\theta} &= \tfrac{1}{2}\left[\frac{1}{(R+z)}\frac{\partial\bar{u}}{\partial\theta} + \frac{\partial\bar{v}}{\partial x}\right] + \frac{1}{2(R+z)}\left[\frac{\partial\bar{u}}{\partial x}\frac{\partial\bar{u}}{\partial\theta} + \frac{\partial\bar{v}}{\partial x}\left(\frac{\partial\bar{v}}{\partial\theta} + \bar{w}\right) + \frac{\partial\bar{w}}{\partial x}\left(\frac{\partial\bar{w}}{\partial\theta} - \bar{v}\right)\right]
\end{aligned} \tag{2}$$

and the strain energy of an element of the shell is

$$V = \frac{E}{2(1-\nu^2)}\int_{-L}^{P}\int_0^{2\pi}\int_{-h/2}^{h/2}[\varepsilon_{xx}^2 + \varepsilon_{\theta\theta}^2 + 2\nu\varepsilon_{xx}\varepsilon_{\theta\theta} + + 2(1-\nu)\,\varepsilon_{x\theta}^2](R+z)\,dz\,d\theta\,dx \tag{3}$$

where 2 L is the axial wavelength of the response. Substitution of Eqs. (1) and (2) into Eq.(3) results in an expression for the strain energy as a function of the displacements of the middle surface. An expression for the kinetic energy is similarly derived.

The generalized coordinates for the displacements are the amplitudes of the modes corresponding to the three frequencies of linear free vibration $\Omega_{i,j}^{(k)}$. The mode shapes having axial wavelengths which are integral multiples of $2L$ are

$$(1/R)u_{i,j}^{(k)} = \alpha_{i,j}^{(k)}\cos(i\theta - \bar{\theta})\sin(j\pi x/L - \bar{x}) \tag{4a}$$

$$(1/R)v_{i,j}^{(k)} = \beta_{i,j}^{(k)}\sin(i\theta - \bar{\theta})\cos(j\pi x/L - \bar{x}) \tag{4b}$$

$$(1/R)w_{i,j}^{(k)} = \gamma_{i,j}^{(k)}\cos(i\theta - \bar{\theta})\cos(j\pi x/L - \bar{x}) \tag{4c}$$

where the modes are normalized so that

$$\alpha_{i,j}^{(k)2} + \beta_{i,j}^{(k)2} + \gamma_{i,j}^{(k)2} = 1 \tag{5}$$

In order to avoid the explicit appearance of the phase angles $\bar{\theta}$ and $\bar{x}$, four generalized coordinates are used for each set of values of i, j and k: $c_{i,j}^{(k)}$ for $\bar{\theta} = \bar{x} = 0$, $d_{i,j}^{(k)}$ for $\bar{\theta} = \pi/2$, $\bar{x} = 0$, $f_{i,j}^{(k)}$ for $\bar{\theta} = 0$, $\bar{x} = \pi/2$, and $g_{i,j}^{(k)}$ for $\bar{\theta} = \bar{x} = \pi/2$. The displacements are then expressed as an infinite sequence of the generalized coordinates over the range of values $i,j = 0, 1, 2\ldots$ and $k = 1, 2, 3$.

If the frequency Ω of the transverse excitation

$$P_z = F\cos n\theta\cos(\pi x/L)\cos\Omega t \tag{6}$$

is close to one of the natural frequencies $\Omega_{n,1}^{(k)}$, which is labeled as $\Omega_{n,1}^{(1)}$, the cooresponding generalized coordinate $c_{n,1}^{(1)}$ would be close to resonance in a linear analysis. Because of the coupling effects of the nonlinearities, the generalized coordinates $c_{n\,1}^{(1)}$, $d_{n\,1}^{(1)}$, $f_{n\,1}^{(1)}$, and $g_{n\,1}^{(1)}$, called the principal coordinates, will be much larger than all others. To express this mathematically we introduce the perturbation parameter $|A|$, which is defined so that

$$c_{n\,1}^{(1)} = A,\ (d/d\tau)\,c_{n\,1}^{(1)} = 0 \quad\text{at}\quad \tau \equiv \Omega t - \varphi = 0 \tag{7}$$

where φ is a phase angle to be determined. As the displacements are small compared to R, $|A| \ll 1$. Then the magnitude of the generalized coordinates must fulfill the requirement

$$c_{i,j}^{(k)}, d_{i,j}^{(k)}, f_{i,j}^{(k)}, g_{i,j}^{(k)}\begin{cases} = \text{order}\ (A)\ \text{if}\ i = n, j = k = 1 \\ \leq \text{order}\ (A^2)\ \text{otherwise}\end{cases} \tag{8}$$

To obtain a consistent first approximation to the effects of nonlinearity, all terms in the energy expressions are retained if they are of order (A^4) or larger. An examination of the spatial integrals occurring in these expressions reveals that secondary coordinates $q_{i,j}^{(k)}$ couple with the principal coordinates only if i and $j = 0$ or $2n$; all other secondary coordinates are smaller than order (A^2) and are dropped from the expressions for the displacements. Substitution of these truncated expressions yields equations for the strain and kinetic energies as polynomial functions of the remaining generalized coordinates. The equations of motion are then obtained from Lagrange's equations.

The frequency is shown to satisfy

$$\Omega = \Omega_{n,1}^{(1)}\,[1 + \text{order}\,(A^2)] \tag{9}$$

and the solutions for the principal coordinates† are

$$c_{n,1}^{(1)} = A\cos\tau + \text{order}\,(A^3)$$

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* Assistant Professor, School of Aeronautics, Astronautics, and Engineering Sciences. Member AIAA.

† Only those responses with the same frequency as the excitation are considered here. However, such responses may show regions of instability which correspond to nonsteady, beating oscillations, such as those occuring in Ref. 2.

$$d_{n,1}^{(1)} = B\cos(\tau + \psi) + \text{order}(A^3)$$
$$f_{n,1}^{(1)} = C\cos(\tau + \zeta) + \text{order}(A^3)$$
$$g_{n,1}^{(1)} = D\cos(\tau + \eta) + \text{order}(A^3) \tag{10}$$

The amplitudes B, C, D are of order (A) unless they vanish identically. The equations relating the amplitudes and phase angles to Ω are found by using Eqs.(9) and (10) to solve for the secondary coordinates and then applying the method of harmonic balance to the differential equations for the principal coordinates. The resulting relations involve six numerical coefficients which are tabulated for several values of the parameters.

Many responses representing different combinations of amplitudes and phase angles are found. In one, only the amplitude A differs from zero, so that the nodal lines of transverse displacement and pressure coincide. The other responses represent combinations of waves traveling in the axial and circumferential directions. Only two of these responses have stationary nodal lines, spiraling around the shell in both cases. In this respect, these results resemble those for rings[1] and finite shells[2,3] wherein waves may propagate around the circumference.

A typical amplitude-frequency diagram, partially illustrated by Fig. 1, shows bifurcations between the branches of different responses. To determine which will actually occur, the stability of the responses must be studied. This problem was too lengthy to be pursued here. It is shown in Ref. 4 that the general system is equivalent to a nonlinear, four-degree-of-freedom system. Unfortunately, the stability of the responses of such a system has not yet been studied, but this does lead to a shortcut in the determination of the stability of the symmetric response.

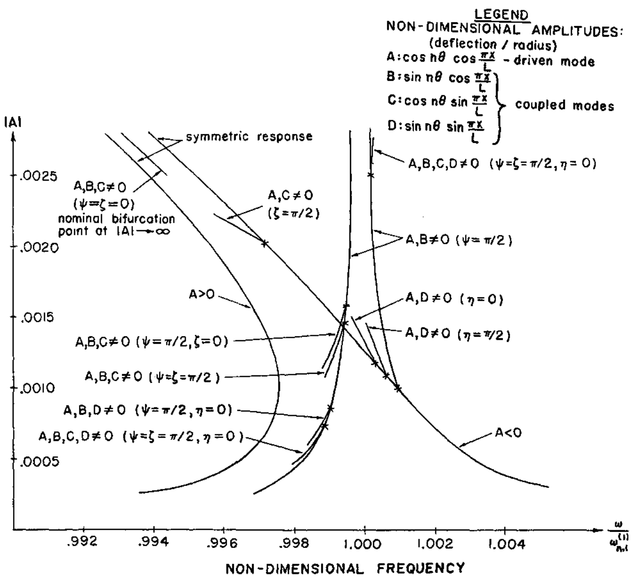

Fig. 1 Composite diagram of all bifurcations from the symmetric response and from the response $A, B = 0$, fundamental frequency $n = 2$, $L/R = 2$, $h/R = 0.01$.

Consideration of effects not included in the solution showed that transverse shear, rotatory inertia, and nonlinear bending were not significant for the tabulated values of the parameter. For a second nonlinear approximation however, these effects are potentially of importance.

The purposes of this paper were mainly to develop the perturbation procedure, in that infinite shells are entirely fictional. The responses of the infinite shell contain vibration modes which are unallowable in finite ones; the coupling between generalized coordinates for the finite shell is different; and waves may not propagate down the axis of the finite shell. There is one important situation where a comparison with previous investigations can be made—the case of plane strain ($L = \infty$). In this case the amplitudes C and D vanish, $\psi \equiv \pi/2$, $\varphi \equiv 0$, and the frequency-amplitude relations are

$$2[\Omega/\Omega_{n,1}^{(1)} - 1] = (P_1 + P_2)A^2 + (P_1 - P_2)B^2 - \bar{F}/A$$
$$B \equiv 0 \quad \text{or} \quad B^2 = A^2 - \bar{F}/2P_2A \tag{11}$$

The values of the numerical coefficients P_1 and P_2 for the case of excitation near the fundamental frequency may be compared to the solution for rings[1], see Table 1. For high values of n there is excellent agreement, but for lower values the results diverge slightly, even though the suggested corrections for the results of Ref. 1 have been made. The present study gives more accurate results, because here all secondary coordinates are retained and a more refined shell theory is utilized.

Table 1 Comparison of coefficients for plane strain
$[\omega_{n,0}^{(1)} = [(1 - \nu^2)\rho R^2/E]^{1/2}\Omega_{n,0}^{(1)}]$

n	Present study $(R/h)\omega_{n,0}^{(1)}$	Present study P_1	Present study P_2	Evensen $P_2(P_1 \equiv 0)$
2	0.775	−0.04	− 0.78	− 1.04
3	2.191	−0.29	− 10.11	− 11.54
4	4.201	−0.70	− 43.47	− 46.89
6	9.966	−1.94	− 273.11	− 282.62
9	22.952	−4.74	−1520.5	−1544.0

References

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